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Perturbative Analysis of a Stationary Magnetosphere in an Extreme Black Hole Spacetime

– On the Meissner-like Effect of an Extreme Black Hole –

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Abstract

It is known that the Meissner-like effect is seen in a magnetosphere without an electric current in black hole spacetime: no non-monopole component of magnetic flux penetrates the event horizon if the black hole is extreme. In this paper, in order to see how an electric current affects the Meissner-like effect, we study a force-free electromagnetic system in a static and spherically symmetric extreme black hole spacetime. By assuming that the rotational angular velocity of the magnetic field is very small, we construct a perturbative solution for the Grad-Shafranov equation, which is the basic equation to determine a stationary, axisymmetric electromagnetic field with a force-free electric current. Our perturbation analysis reveals that, if an electric current exists, higher multipole components may be superposed upon the monopole component on the event horizon, even if the black hole is extreme.

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I. INTRODUCTION

It is widely believed that there are supermassive black holes at the centers of galaxies, and these are hypothesized to be the central engines for active galactic nuclei (AGNs) and gamma ray bursts (GRBs). Two main possibilities are considered as the energy source. One is the gravitational energy of accreting matter and the other is the rotational energy of the black hole or the accretion disk surrounding it. However, the details of the energy extraction process are not clear. It is also not understood well how the energy is converted into that of AGNs or GRBs.

Blandford and Znajek showed that the rotational energy of a rotating black hole can be extracted in the form of Poynting flux along magnetic field lines penetrating the event horizon [1], which is known as the Blandford-Znajek (BZ) mechanism. Its efficiency depends on the rotational velocity of the black hole and the configuration of the magnetic field: the extraction of the rotational energy becomes more efficient the more magnetic field lines penetrates the event horizon and the more rapidly the black hole rotates. In the BZ mechanism, poloidal magnetic fields which penetrate the event horizon play a crucial role for the energy extraction as well as for the formation of jets associated with AGNs. In fact, some numerical studies reported that Poynting-dominated jets were produced [2–4].

Bičák and Janiš showed that a magnetic field without an electric current is expelled from the event horizon of a maximally rotating black hole [5]. This is analogous to the Meissner effect in a superconductor. This effect for a rapidly rotating black hole would decrease the efficiency of the BZ mechanism, though the larger rotational velocity of the black hole would increase the efficiency. In realistic astrophysical cases, however, there would be plasma around the black hole. How the Meissner-like effect is affected by the existence of plasma is the main subject of this paper. We clarify the effect of an electric current on the Meissner-like effect of an extreme black hole. Komissarov and McKinney studied numerically the Meissner-like effect of a Kerr black hole [6]. They carried out numerical simulations for a highly conductive magnetosphere until it almost reaches steady state, and there was no sign of the Meissner-like effect in their numerical results. In this paper, we study how an electric current affects the Meissner-like effect by solving a stationary problem analytically.

Since realistic situations are, in general, very complicated, it is difficult to model them. In order to reveal the essence of the plasma effect, we consider a very simple toy model:

(i) we consider a stationary, axisymmetric force-free system of the electromagnetic field and plasma; (ii) we consider a static spherically symmetric black hole spacetime with a degenerate horizon as a background spacetime rather than a rotating black hole. The degenerate horizon is the origin of the Meissner-like effect in a vacuum black hole spacetime [7], and hence, by studying the electromagnetic field in this spacetime, we can see whether the Meissner-like effect remains even in the case with an electric current. The spacetime considered in this paper is known as the Reissner-Nordström (RN) spacetime. By these assumptions, the basic equations reduce to only one quasi-linear elliptic equation for the magnetic flux function called the Grad-Shafranov (GS) equation [8].

For the black hole spacetime, the GS equation has three regular singular points: one is at the event horizon, and the other two are at the inner and outer light surfaces on which the velocities of the magnetic field lines agree with the speed of light. For non-extreme cases, one boundary condition is imposed at each regular singular point so that the magnetic field is smooth everywhere. However, for a given electric current function, the obtained solution for the magnetic flux need not be C^1 but at most C^{1-} [9]. Although numerical C^1 solutions have been obtained by iteratively changing the functional form of the electric current [9–14], a mathematically rigorous proof for the existence of a C^1 solution has not yet been presented. Furthermore, in the extreme case, two kinds of boundary condition must be imposed at once on the event horizon. We shall mention all these difficulties in solving the GS equation in §IV.

As will be shown in §V, the monopole component is a unique configuration of the magnetic field on the event horizon if there is not an electric current. Since there is no magnetic monopole in nature, this result implies the Meissner-like effect of the extreme RN black hole. In order to study the electromagnetic field coupled to an electric current around an RN black hole, we use a perturbative method which includes two expansion parameters. One of these parameters corresponds to the rotational angular velocity of the magnetic fields. Namely, we consider slow-rotating magnetic fields as was first considered by Blandford and Znajek [1]. The other parameter is the ratio of the distance from the event horizon to the horizon radius, since we consider only the vicinity of the event horizon, which includes the inner light surface. Although we cannot take into account the outer light surface in our perturbative method, we can obtain approximate solutions sufficient to study the Meissner-like effect with an electric current.

This paper is organized as follows. In §II, we introduce the RN black hole as a background geometry. Then we show the GS equation for the RN spacetime in §III; the detailed derivation of the GS equation is given in Appendices A and B. The regularity conditions for the GS equation and difficulties in solving this equation are described in detail in §IV. Using perturbative analyses, we study the cases with and without an electric current in §V and VI, respectively. §VII is devoted to summary and discussion. In Appendix C, we show the relation between the Kerr-Schild coordinate system and the standard static coordinate system of the RN spacetime. In Appendix D, we give a proof of a theorem on the magnetic field obtained by the present perturbative method.

In this paper, we adopt the geometrized units, in which the Newton's gravitational constant and the speed of light are unity, and the abstract index notation: small Latin indices, excluding t and r, indicate the type of tensor, whereas small Greek indices, excluding θ and φ, represent components with respect to the coordinate basis. The exceptional indices t, r, θ, and φ denote the components of time, and the radial and azimuthal coordinates in the spherical polar coordinate system. The signature of the metric is diag$[-,+,+,+]$.

II. BACKGROUND GEOMETRY

We consider a static and spherically symmetric spacetime of the following metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -\alpha^2 dt^2 + \frac{r^2}{\Delta}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \tag{2.1}$$

with

$$\Delta = (r-r_+)(r-r_-) \qquad \text{and} \qquad \alpha = \frac{\sqrt{\Delta}}{r}, \tag{2.2}$$

where we assume $r_+ \geq r_- > 0$. This spacetime is known as the RN spacetime. There are two horizons, which are determined by $\Delta = 0$: r_+ and r_- represent the radius of the event and Cauchy horizons, respectively. The case of $r_+ = r_- := r_{\rm H}$ is called the extreme case.

III. GRAD-SHAFRANOV EQUATION

Maxwell's equations are given by

$$\nabla_{[a}F_{bc]} = 0, \tag{3.1}$$

$$\nabla_b F^{ab} = 4\pi J^a, \tag{3.2}$$

where F_{ab} is the field strength tensor of the electromagnetic field, J^a is a current density, and ∇_b is the covariant derivative[1]. If the field strength tensor is expressed by using a 4-vector potential A_a as

$$F_{ab} = \nabla_a A_b - \nabla_b A_a = \partial_a A_b - \partial_b A_a, \tag{3.3}$$

then Eq. (3.1) is trivially satisfied, where ∂_a is the ordinary derivative.

As mentioned in §I, hereafter, we consider the axisymmetric and stationary electromagnetic field in the RN spacetime. In order to make the problem simple, we assume that the system field satisfies the force-free condition

$$F_{ab} J^b = 0. \tag{3.4}$$

Hereafter, we focus on the system of only Eqs. (3.2) and (3.4). The formulation of a force-free electrodynamics field in the black hole spacetime was given by Macdonald and Thorne [8]. Our formulation is based on their work.

In the case of a stationary, axisymmetric electromagnetic field, we can define the "angular velocity" of the magnetic field as

$$\Omega_{\mathrm{F}} := \frac{F_{t\theta}}{F_{\theta\varphi}} = \frac{F_{tr}}{F_{r\varphi}}. \tag{3.5}$$

The reason why Ω_{F} can be regarded as the angular velocity of the magnetic field is described in Appendix A. From Eqs. (3.2) and (3.4), the GS equation is obtained as

$$\Psi'' + \frac{1}{\Delta} L_\theta \Psi + \frac{U}{D} + \frac{W}{\Delta D} = 0, \tag{3.6}$$

where a prime $'$ represents a derivative with respect to r,

$$L_\theta \Psi = \sin\theta \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right), \tag{3.7}$$

$$U = \left(D' + \frac{r^2}{2}\sin^2\theta \frac{d\Omega_{\mathrm{F}}^2}{d\Psi}\Psi'\right)\Psi', \tag{3.8}$$

$$W = \left(\partial_\theta D + \frac{r^2}{2}\sin^2\theta \frac{d\Omega_{\mathrm{F}}^2}{d\Psi}\partial_\theta\Psi\right)\partial_\theta\Psi + 8\pi^2 r^2 \frac{dI^2}{d\Psi}, \tag{3.9}$$

[1] We only use the RN spacetime as a background geometry rather than the Kerr spacetime, that is, a test U(1) field on the RN spacetime is considered in our toy model. Bičák and Dvořák studied perturbations of the coupled Einstein-Maxwell system on the extreme RN spacetime without an electric current [15]. The Meissner-like effect appeared in this case.

where $I = I(\Psi)$ and Ψ are the electric current and the magnetic flux through an axisymmetric polar cap, which are defined by Eqs. (B4) and (B5), respectively, and D is defined by

$$D := \alpha^2 - r^2\Omega_{\rm F}^2 \sin^2\theta. \tag{3.10}$$

The derivation of the GS equation (3.6) is given in Appendices A and B.

In general, the GS equation for a black hole magnetosphere has three regular singular points: one is at the event horizon $\Delta = 0$, and the other two are at the light surfaces defined by $D = 0$. A light surface is a timelike hypersurface on which the rotational speed of magnetic field lines are equal to the speed of light. The inner light surface given by a function $r = r_{\rm LS-}(\theta)$ has a spacelike section with spherical topology, whereas the outer one, $r = r_{\rm LS+}(\theta)$, has that of cylindrical topology.

In the case of the Kerr spacetime, the Kerr-Schild coordinate system is often adopted, for example, in numerical simulations (e.g., [2, 16, 17]), since there is no coordinate singularity on the event horizon. Therefore, we give the Kerr-Schild coordinate system for the RN spacetime in Appendix C. In Appendix C, we show that the singular point on the event horizon in the GS equation appears even if we adopt Kerr-Schild coordinates. This is also true in the case of the Kerr spacetime, though it is not shown in this paper (we will show it elsewhere). As long as a stationary magnetic field is considered, the singular point of the basic equation will appear on the event horizon, since there is no timelike Killing vector field on or inside the event horizon, or in other words, the stationary configuration cannot be realized inside the black hole.

IV. REGULARITY CONDITIONS AND BOUNDARY CONDITIONS

In this section, we consider only the case of $\Omega_{\rm F} \neq 0$. Then, by virtue of the symmetry of the background spacetime, without loss of generality, we may assume $\Omega_{\rm F} > 0$. The case of $\Omega_{\rm F} = 0$ will be treated as specific cases later.

A. Symmetry Axis $\theta = 0$

On the symmetry axis $\theta = 0$, both Ψ and I should vanish by their definitions. I is a function of Ψ, and thus, $I|_{\Psi=0}$ should vanish.

B. Event Horizon

In Appendix C, we give the components of F_{ab} in the Kerr-Schild coordinate system (T, Φ, R, Θ). From Eq. (C13), we have

$$F_{R\Theta} = -\frac{\mathcal{M}}{\Delta}, \tag{4.1}$$

where

$$\mathcal{M} = \frac{1}{2\pi}\left(2Mr - r_+^2\right)\Omega_{\mathrm{F}}\partial_\theta\Psi + \frac{2r^2}{\sin\theta}I(\Psi). \tag{4.2}$$

Since the Kerr-Schild coordinate system is non-singular on the event horizon, $F_{R\Theta}$ must be finite there. Thus, we have $\mathcal{M}|_{r=r_+} = 0$. This leads to

$$I + \frac{1}{4\pi r^2}(2Mr - r_+^2)\Omega_{\mathrm{F}}\sin\theta\partial_\theta\Psi = 0 \qquad \text{at} \quad r = r_+ \ . \tag{4.3}$$

The above condition corresponds to the horizon boundary condition derived by Znajek for the Kerr black hole [18]. It is seen from the GS equation (3.6) that, in order that Ψ, Ψ' and Ψ'' are finite on the event horizon $r = r_+$, the following condition should be satisfied

$$L_\theta\Psi + \frac{W}{D} = 0. \tag{4.4}$$

However, this condition is satisfied if Ψ satisfies the regularity condition (4.3), and thus no additional constraint is imposed by this equation.

In the extreme case, since the equation $\Delta = 0$ has a double root $r_\pm = r_{\mathrm{H}}$, $\mathcal{M}'|_{r=r_{\mathrm{H}}}$ should vanish as well as $\mathcal{M}|_{r=r_{\mathrm{H}}} = 0$ so that $F_{R\Theta}$ is finite on the event horizon. These conditions imply

$$I + \frac{\Omega_{\mathrm{F}}}{4\pi}\sin\theta\partial_\theta\Psi \ = \ 0, \tag{4.5}$$

$$\frac{dI}{d\Psi}\Psi' + \frac{\sin\theta}{4\pi}\left(\Omega_{\mathrm{F}}\partial_\theta\Psi' + \frac{d\Omega_{\mathrm{F}}}{d\Psi}\Psi'\partial_\theta\Psi\right) \ = \ 0, \tag{4.6}$$

on the event horizon $r = r_{\mathrm{H}}$. We can see from the GS equation (3.6) that, in order that Ψ, Ψ', and Ψ'' are finite on the event horizon for the extreme case, not only Eq. (4.4) but also the following condition should be satisfied,

$$\left(L_\theta\Psi + \frac{W}{D}\right)' = 0 \quad \text{at} \quad r = r_{\mathrm{H}}. \tag{4.7}$$

This condition is satisfied if Ψ satisfies the regularity conditions (4.5) and (4.6), and thus no additional constraint is imposed by this equation.

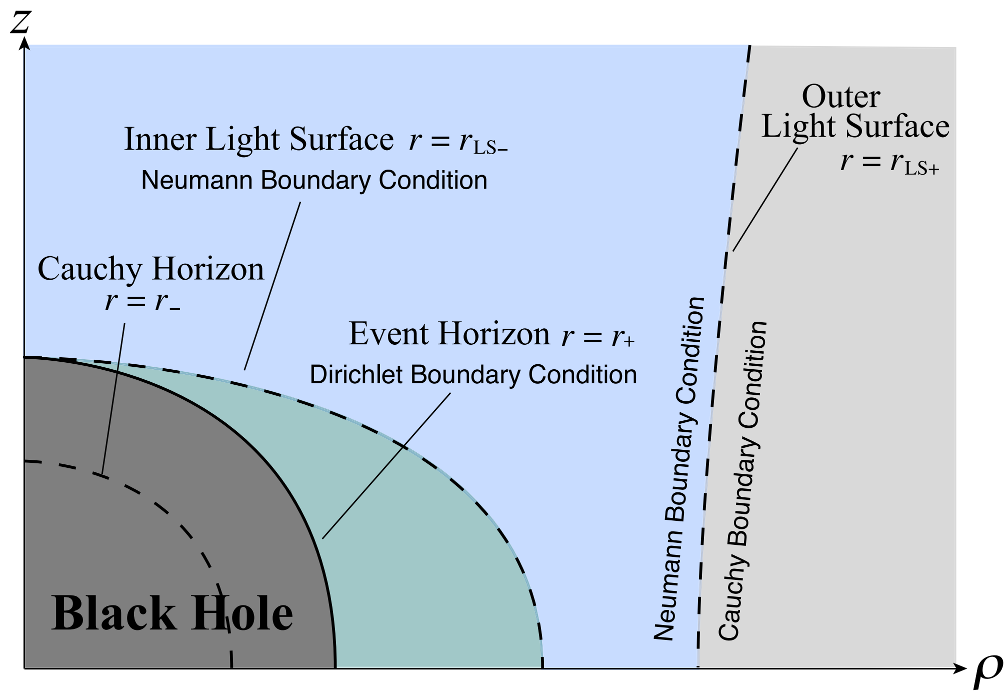

FIG. 1: Schematic diagram of the magnetosphere of a non-extreme RN black hole $r_+ > r_-$, where $\rho = r\sin\theta$ and $z = r\cos\theta$.

C. Light Surface

As mentioned, the light surfaces are singular points of the GS equation (3.6). In the extreme case, the radial coordinates of the light surfaces, which are the roots of the equation $D = 0$ in the domain $r > r_{\mathrm{H}}$, are given by

$$r = r_{\mathrm{LS}\pm} = \frac{1 \pm \sqrt{1 - 4r_{\mathrm{H}}\Omega_{\mathrm{F}}\sin\theta}}{2\Omega_{\mathrm{F}}\sin\theta}. \tag{4.8}$$

In order that Ψ'' and $L_\theta\Psi$ are finite on the light surfaces, $U + W/\Delta$ must vanish there. This requirement leads to the following regularity conditions on the light surfaces:

$$V^a\partial_a\Psi = -8\pi^2 r^2 \frac{dI^2}{d\Psi} \qquad \text{at} \quad r = r_{\mathrm{LS}\pm} \tag{4.9}$$

where

$$V^a := r^2 g^{ab}\left(\partial_b D + \frac{r^2}{2}\sin^2\theta\,\partial_b\Omega_{\mathrm{F}}^2\right). \tag{4.10}$$

D. Boundary Conditions in the Case of Non-Degenerate Horizons

For simplicity, in this subsection, we assume that $\Omega_{\rm F}$ is constant. By virtue of this assumption, the conditions (4.9) become Neumann boundary conditions on the light surfaces.

Here, we consider the non-extreme case $r_+ > r_-$. Let us assume that the functional form of $I(\Psi)$ has already been determined before solving the GS equation. Then, by solving the horizon regularity condition (4.3), we obtain Ψ on the event horizon. Since Ψ is the magnetic flux through the polar cap (see Appendix B), Ψ should vanish at $\theta = 0$. Thus, there seems to be no freedom for setting a boundary condition for Ψ in solving Eq. (4.3), but this is not true. Since $\theta = 0$ is a regular singular point of Eq. (4.3), there still remains one degree of freedom for choosing a boundary value of the second-order derivative of Ψ. Hence, a Dirichlet boundary condition for the GS equation is determined on the event horizon by the regularity condition (4.3). By imposing this Dirichlet boundary condition at $r = r_+$ and further Neumann boundary conditions at $r = r_{\rm LS-}$ (4.9) and on the equatorial plane (e.g., the reflection-symmetric boundary condition $\partial_\theta \Psi|_{\theta=\pi/2} = 0$), a solution for the GS equation (3.6) is uniquely determined in the domain $r_+ < r < r_{\rm LS-}$. By imposing two Neumann boundary conditions (4.9) on the two light surfaces $r = r_{\rm LS\pm}$ and the equatorial plane $\theta = \pi/2$, and a further Dirichlet boundary condition $\Psi = 0$ on the symmetry axis $\theta = 0$, a solution for the GS equation is uniquely determined in the domain $r_{\rm LS-} < r < r_{\rm LS+}$. For the domain $r_{\rm LS+} < r < \infty$, if we impose a boundary condition at Ψ for $r \to \infty$, then we can obtain a solution for the GS equation.

The GS equation can be solved for these three domains, $r_+ < r < r_{\rm LS-}$, $r_{\rm LS-} < r < r_{\rm LS+}$, and $r_{\rm LS+} < r < \infty$, independently by the above procedure (see Fig. 1). Thus, for an arbitrary electric current $I(\Psi)$, the obtained solution for Ψ is, in general, not C^1 but at most C^{1-} at the boundaries $r = r_{\rm LS\pm}$. It was first reported by Contopoulos, Kazanas, and Fendt (CKF) that the continuity of the first-order derivative of Ψ at the light surface as well as the continuity of Ψ itself determines the functional form of the electric current $I(\Psi)$ in the case of the pulsar magnetosphere [9]. They numerically obtained C^1 solutions for Ψ by an iterative method in which both Ψ and the functional form of $I(\Psi)$ are determined simultaneously. The CKF method was used by several authors to study the pulsar magnetosphere and they showed that this method also suitable for their case of interest [10, 13, 14]. However, it should be noted that a mathematically rigorous proof for the existence of the C^1 solution

for the GS equation has not yet been given.

Although there is at most one light surface in the case of the pulsar magnetosphere, there can be two light surfaces if $\Omega_{\rm F}$ is a non-vanishing constant in the case of the black hole magnetosphere (see Fig. 1). Thus, if we chose the functional form of $I(\Psi)$ such that Ψ is C^1 in the domain $r_+ < r < r_{\rm LS+}$, imposing an asymptotic boundary condition for $r \to \infty$ does not guarantee the continuity of the derivative of Ψ at the outer light surface $r = r_{\rm LS+}$. Thus, in order to obtain a solution which is C^1 in the domain $r_+ < r < \infty$, we need to solve the GS equation for the outermost domain $r_{\rm LS+} < r < \infty$ as a Cauchy problem with a boundary data for Ψ and the derivatives of Ψ on $r = r_{\rm LS+}$. This implies that we cannot impose the asymptotic boundary condition for $r \to \infty$. In general, it is difficult to solve an elliptic-type differential equation numerically, such as the GS equation, as a Cauchy problem due to the numerical instability. Thus, in the case with two light surfaces, it is difficult to numerically obtain a solution for the GS equation in the outermost domain $r_{\rm LS+} < r < \infty$. However, this might not be a serious problem, since we may understand whether the Blandford-Znajek mechanism works by studying only the domain $r_+ < r < r_{\rm LS+}$.

Uzdensky applied the CKF method to the magnetospheres of the Schwarzschild black hole [11] and of the Kerr black hole [12], but he focused on only the cases in which there is only one light surface by virtue of a particular assumption on $\Omega_{\rm F}$: Uzdensky assumed that $\Omega_{\rm F}$ asymptotically decreases, and hence there is no outer light surface. Thus, Uzdensky succeeded in numerically obtaining global solutions without solving Cauchy problems for the GS equation.

E. Boundary Conditions in the Case of a Degenerate Horizon

Here, we consider the extreme case $r_+ = r_- = r_{\rm H}$, which is the main case of interest in this paper. In this subsection, we also assume that $\Omega_{\rm F}$ is constant. The horizon regularity conditions (4.5) and (4.6) give boundary values of Ψ and the derivative of Ψ. Thus, in the extreme case, we must solve the GS equation (3.6) as a Cauchy problem even for the domain $r_{\rm H} < r < r_{\rm LS-}$ (see Fig. 2). As mentioned, it is difficult to numerically solve the GS equation as a Cauchy problem, and hence, it seems to be difficult to numerically obtain a solution for the physically important domain $r_{\rm H} < r < r_{\rm LS+}$ in the extreme case. Further, even if we find a procedure for numerically solving the GS equation as a Cauchy problem, the

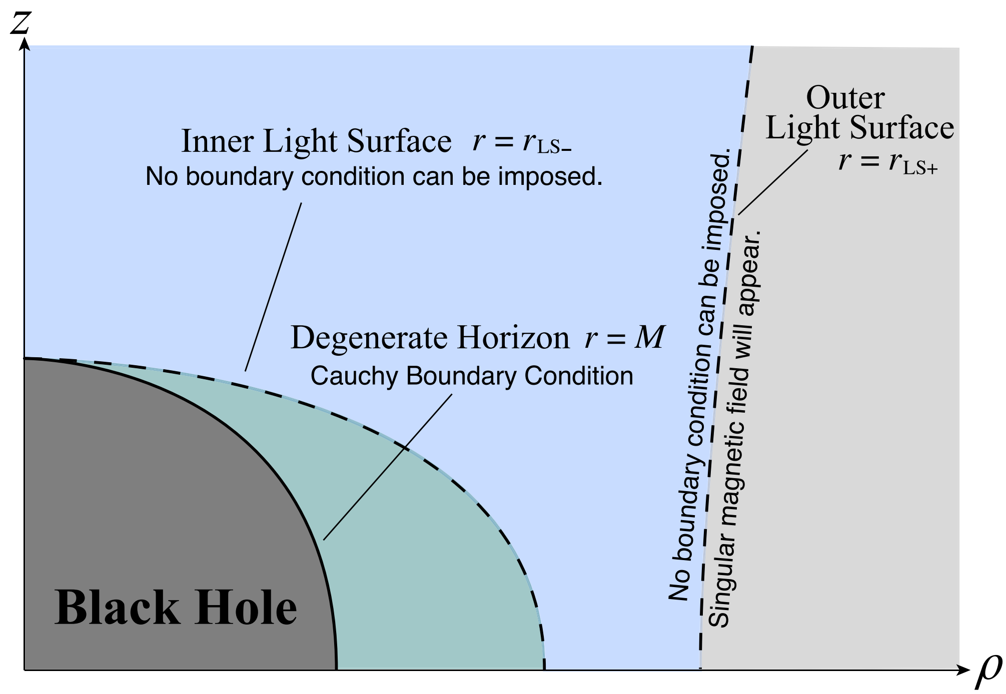

FIG. 2: The same as Fig. 1, but for an extreme RN black hole $r_+ = r_- = r_{\mathrm{H}}$.

regularity condition on the inner light surface $r = r_{\mathrm{LS}-}$ may not be satisfied for an arbitrary functional form of electric current $I(\Psi)$: we must assume the functional form of $I(\Psi)$ to solve the GS equation as a Cauchy problem, but, in general, the assumed electric current $I(\Psi)$ does not satisfy the regularity condition on the inner light surface. As a result, it seems to be impossible to obtain a solution for the GS equation numerically, which is finite on the inner light surface, in the extreme case. In this sense, the perturbative analytic approach discussed in §VI is very important.

We should note that even if we find analytically the electric current I which guarantees the finiteness of Ψ and its derivative on the inner light surface, such a electric current I might not guarantee the finiteness of both Ψ and its derivative on the outer light surface. This implies that either the force-free condition should break down near the outer light surface or the rotational velocity should decay far from the black hole so that the outer light surface does not exist, as in the situation studied by Uzdensky.

V. VACUUM CASE

In this section, we consider the vacuum case $I = 0$.

A. $\Omega_{\rm F} = 0$ on the horizon

Here, we consider the case of $\Omega_{\rm F} = 0$ on the event horizon, $r = r_+$. Even if $\Omega_{\rm F}$ does not vanish except on the event horizon, it satisfies

$$\partial_\theta \Omega_{\rm F} = \frac{d\Omega_{\rm F}}{d\Psi} \partial_\theta \Psi = 0 \qquad \text{at} \ \ r = r_+. \tag{5.1}$$

Thus, on the event horizon, $\partial_\theta \Psi = 0$ or $d\Omega_{\rm F}/d\Psi = 0$ should hold. In the former case, Ψ vanishes on the event horizon, implying that the magnetic flux does not penetrate the event horizon. In the latter case, from Eq. (3.6), in order that Ψ, Ψ', and Ψ'' are finite on the event horizon, the following equation should be satisfied:

$$L_\theta \Psi + r^2 \Psi' \left(\frac{\Delta}{r^2} \right)' = 0 \qquad \text{at} \ \ r = r_+. \tag{5.2}$$

In the extreme case, since $\Delta' = 0$ also holds on the event horizon $r = r_{\rm H}$, we have

$$L_\theta \Psi = 0 \qquad \text{at} \ \ r = r_{\rm H}. \tag{5.3}$$

The solution of the above equation which satisfies the regularity condition on the symmetry axis $\theta = 0$ is

$$\Psi = C(1 - \cos\theta), \tag{5.4}$$

where C is an integration constant. The above solution implies that the magnetic field which can penetrate the event horizon is the only monopole component. By contrast to the extreme case, non-monopole components can penetrate the event horizon in the non-extreme case, since, in this case, Eq. (5.2) does not necessarily imply $L_\theta \Psi = 0$. As a result, we can conclude that the Meissner-like effect of the extreme black hole appears in the vacuum case.

In the case that $\Omega_{\rm F}$ vanishes everywhere, we can obtain global solutions. The solutions which satisfy the regularity condition on the symmetry axis $\theta = 0$ are written in the form

$$\Psi = \sum_{l=0}^{\infty} R_l(r) P_l^1(\cos\theta) \sin\theta, \tag{5.5}$$

where $P_l^1(x)$ is the associated Legendre function of the first kind with $m = 1$. Then, the GS equation (3.6) becomes

$$\alpha^2 R_l'' + 2\alpha' \alpha R_l' - \frac{l(l+1)}{r^2} R_l = 0. \tag{5.6}$$

In the extreme case, in order that R_l, R_l', and R_l'' are finite on the event horizon, R_l of $l \geq 1$ vanishes, and the only monopole component $l = 0$ may remain.

B. $\Omega_{\rm F} \neq 0$ on the horizon

In this case, from the regularity condition on the event horizon (4.3), we have

$$\partial_\theta \Psi = 0, \tag{5.7}$$

for both extreme and non-extreme cases. Thus, $\Psi = 0$ is a solution which satisfies the regularity condition on the symmetry axis $\theta = 0$. The magnetic field does not penetrate the event horizon at all.

VI. CASE WITH AN ELECTRIC CURRENT

In this section, we focus on the extreme case $r_\pm = r_{\rm H}$ and assume that the rotational velocity of the magnetic field $\Omega_{\rm F}$ is constant.

In the case of $\Omega_{\rm F} = 0$, it is seen from Eq. (3.6) that since $D = \alpha^2$, the following condition should be satisfied on the event horizon:

$$\frac{dI^2}{d\Psi} = 0. \tag{6.1}$$

The above condition allows $I =$ const. on the event horizon. However, since I should vanish on the symmetry axis $\theta = 0$, the allowed constant is zero. Thus, in this case, the same argument as was used in the vacuum case discussed in the previous section is also true. The allowed configuration of the magnetic field on the horizon is only the monopole component (5.4). Hence, hereafter, we focus on the case of $\Omega_{\rm F} \neq 0$.

A. Grad-Shafranov equation near the event horizon

We are interested in the configuration of the magnetic field near the event horizon. In order to analyze the GS equation, we introduce the following dimensionless quantities:

$$\begin{aligned}
r &=: r_{\rm H} y, & (6.2)\\
\Psi &=: r_{\rm H} \psi, & (6.3)\\
\varepsilon &:= r_{\rm H} \Omega_{\rm F}, & (6.4)\\
I(\Psi) &=: \varepsilon \mathcal{I}(\psi), & (6.5)\\
8\pi^2 \frac{dI^2}{d\Psi} &=: \varepsilon^2 r_{\rm H}^{-1} \mathcal{S}(\psi). & (6.6)
\end{aligned}$$

Using these quantities, the GS equation (3.6) becomes

$$\partial_y^2\psi + \frac{1}{(y-1)^2}L_\theta\psi + \frac{\mathcal{U}}{\mathcal{D}} + \frac{\mathcal{W}}{(y-1)^2\mathcal{D}} = 0, \tag{6.7}$$

where

$$\begin{aligned} \mathcal{D} &= y\left[(y-1)^2 - \varepsilon^2 y^4 \sin^2\theta\right], & (6.8)\\ \mathcal{U} &= 2(y-1-\varepsilon^2 y^4 \sin^2\theta)\partial_y\psi, & (6.9)\\ \mathcal{W} &= -\varepsilon^2 y^5\left[\sin 2\theta\partial_\theta\psi - \mathcal{S}(\psi)\right]. & (6.10) \end{aligned}$$

The regularity conditions on the event horizon (4.5) and (4.6) become

$$\begin{aligned} \mathcal{I}(\psi) + \frac{1}{4\pi}\sin\theta\partial_\theta\psi &= 0, & (6.11)\\ \frac{d\mathcal{I}}{d\psi}\partial_y\psi + \frac{1}{4\pi}\sin\theta\partial_\theta(\partial_y\psi) &= 0. & (6.12) \end{aligned}$$

The regularity conditions on the light surfaces (4.9) become

$$\partial_\theta\psi - \frac{\mathcal{S}(\psi)}{\sin 2\theta} - \varepsilon y\tan\theta\sin\theta(1-\varepsilon y^2\sin\theta)\partial_y\psi = 0 \qquad \text{at} \;\; y = y_{\mathrm{LS}\pm}, \tag{6.13}$$

where

$$y_{\mathrm{LS}\pm} := \frac{r_{\mathrm{LS}\pm}}{r_{\mathrm{H}}}. \tag{6.14}$$

Here, we assume that ψ can be written in the form of Taylor series around the event horizon, $y = 1$, as

$$\psi = \sum_{n=0}^{\infty}\psi^{(n)}(\theta)(y-1)^n. \tag{6.15}$$

The coefficients $\psi^{(n)}$ are, in principle, determined by the GS equation (6.7) with the regularity conditions (6.11), (6.12), and (6.13). Using the expression (6.15), Eqs. (6.11) and (6.12) can be rewritten in the forms

$$\begin{aligned} \mathcal{I}(\psi^{(0)}) + \frac{1}{4\pi}\sin\theta\frac{d\psi^{(0)}}{d\theta} &= 0, & (6.16)\\ \frac{d\mathcal{I}}{d\psi}(\psi^{(0)})\psi^{(1)} + \frac{1}{4\pi}\sin\theta\frac{d\psi^{(1)}}{d\theta} &= 0. & (6.17) \end{aligned}$$

If we fix the functional form of $\mathcal{I}(\psi)$, we obtain $\psi^{(0)}$ and $\psi^{(1)}$ from the above equations, and further, we obtain $\psi^{(n)}$ of $n \geq 2$ from Eq. (6.7); in order to get $\psi^{(n)}$ for $n \geq 3$, we use

an equation obtained by $(n-2)$-times differentiation of Eq. (6.7) with respect to y. For example, for $n=2$, by evaluating the GS equation (6.7) on the event horizon $y=1$, we have

$$\begin{aligned}&\varepsilon^2\left[L_\theta\psi^{(2)}+2\psi^{(2)}+2\cot\theta\left(\frac{d\psi^{(2)}}{d\theta}-\frac{1}{\sin 2\theta}\frac{d\mathcal{S}}{d\psi}(\psi^{(0)})\psi^{(2)}\right)\right]\\&=-2\cot\theta\left(\frac{d\psi^{(0)}}{d\theta}-\frac{\mathcal{S}(\psi^{(0)})}{\sin 2\theta}\right)\left(\frac{1}{\sin^2\theta}+12\varepsilon^2\right)-\varepsilon^2\psi^{(1)}\left[2-\frac{1}{2\sin^2\theta}\frac{d^2\mathcal{S}}{d\psi^2}(\psi^{(0)})\psi^{(1)}\right].\end{aligned} \tag{6.18}$$

Here, we should again note that $\mathcal{I}(\psi)$ cannot be freely specified. The functional form of $\mathcal{I}(\psi)$ must be chosen such that the regularity condition (6.13) on the inner light surface is satisfied.

B. Perturbative analysis

We consider slowly rotating magnetic fields, or in other words, we assume $0<\varepsilon\ll 1$. We rewrite the basic equations in the form of the power series with respect to ε, and then we construct a solution of ψ on the horizon, i.e., $\psi^{(0)}$, by perturbative procedures with respect to ε. Although, as mentioned, it seems to be impossible to determine the functional form of $\mathcal{I}$ numerically, we can find it by this method.

In order to construct a perturbative solution for $\psi^{(n)}$, we write

$$\psi^{(n)}(\theta)=\sum_{N=0}^{\infty}\psi_N^{(n)}\varepsilon^N. \tag{6.19}$$

Further, we assume

$$\mathcal{I}(x)=\sum_{N=0}^{\infty}\mathcal{I}_{N+1}(x)\varepsilon^N \quad\text{and}\quad \mathcal{S}(x)=\sum_{N=0}^{\infty}\mathcal{S}_{N+2}(x)\varepsilon^N. \tag{6.20}$$

From Eq. (4.8), the location of the inner light surface is written as

$$y_{\mathrm{LS}-}=1+\sum_{n=1}^{\infty}\frac{(2n)!}{n!(n+1)!}(\varepsilon\sin\theta)^n=1+\varepsilon\sin\theta+2(\varepsilon\sin\theta)^2+\cdots. \tag{6.21}$$

Because $y_{\mathrm{LS}-}-1=\mathcal{O}(\varepsilon)$, we can express the quantities on the inner light surface by using the quantities on the event horizon. For example, $\partial_\theta\psi$ at $y=y_{\mathrm{LS}-}$ is written as

$$\partial_\theta\psi(y_{\mathrm{LS}-},\theta)=\frac{d\psi_0^{(0)}}{d\theta}+\varepsilon\left(\frac{d\psi_1^{(0)}}{d\theta}+\sin\theta\frac{d\psi_0^{(1)}}{d\theta}\right)+\cdots. \tag{6.22}$$

Since the main purpose of this study is to see the effect of an electric current on the configuration of the magnetic field on the event horizon, we focus on $\psi^{(0)}$. For this purpose, we rewrite Eqs. (6.17) and (6.18) in more appropriate forms as follows.

By differentiating Eq. (6.16) with respect to θ, we have

$$4\pi \frac{d\mathcal{I}}{d\psi}(\psi^{(0)}) + \left(\frac{d\psi^{(0)}}{d\theta}\right)^{-1} \frac{d}{d\theta}\left(\sin\theta \frac{d\psi^{(0)}}{d\theta}\right) = 0. \tag{6.23}$$

From Eq. (6.17), we have

$$4\pi \frac{d\mathcal{I}}{d\psi}(\psi^{(0)}) + \sin\theta \frac{d}{d\theta} \ln \psi^{(1)} = 0. \tag{6.24}$$

By subtracting Eq. (6.23) from the above equation, we obtain

$$\left(\frac{d\psi^{(0)}}{d\theta}\right)^{-1} \frac{d}{d\theta}\left(\sin\theta \frac{d\psi^{(0)}}{d\theta}\right) - \sin\theta \frac{d}{d\theta} \ln \psi^{(1)} = 0. \tag{6.25}$$

It is easy to integrate the above equation, and we have

$$\psi^{(1)} = C' \sin\theta \frac{d\psi^{(0)}}{d\theta}, \tag{6.26}$$

where C' is an integration constant. In order to obtain $\psi^{(1)}$, we use Eq. (6.26) rather than Eq. (6.17).

By substituting Eq. (6.16) into Eq. (6.23) and using Eqs. (6.5) and (6.6), we have

$$-2\cot\theta \left(\frac{d\psi^{(0)}}{d\theta} - \frac{\mathcal{S}(\psi^{(0)})}{\sin 2\theta}\right) = L_\theta \psi^{(0)}. \tag{6.27}$$

The above equation is equivalent to Eq. (4.4). By substituting Eq. (6.27) into the first term on the right-hand side of Eq. (6.18), we obtain

$$\begin{aligned} L_\theta \psi^{(0)} \;=\; & \varepsilon^2 \sin^2\theta \Bigg[-12 L_\theta \psi^{(0)} + L_\theta \psi^{(2)} + 2\psi^{(2)} \\ & +2\cot\theta \left(\frac{d\psi^{(2)}}{d\theta} - \frac{1}{\sin 2\theta}\frac{d\mathcal{S}}{d\psi}\psi^{(2)}\right) + \psi^{(1)}\left(2 - \frac{1}{2\sin^2\theta}\frac{d^2\mathcal{S}}{d\psi^2}\psi^{(1)}\right)\Bigg]. \end{aligned} \tag{6.28}$$

We shall use the above equation rather than Eq. (6.18).

1. *Zeroth-Order Solutions for* $\psi^{(0)}$

Here, we obtain the zeroth-order solutions for $\psi^{(0)}$. Hereafter, the arguments of $\mathcal{S}_N$ and $\mathcal{I}_N$ are $\psi_0^{(0)}$ as long as we do not specify them.

First of all, we write down the equations to obtain the zeroth-order solutions for $\psi^{(0)}$. From the lowest order of Eqs. (6.13), (6.16), and (6.28), we have

$$\begin{aligned} \frac{d\psi_0^{(0)}}{d\theta} - \frac{\mathcal{S}_2}{\sin 2\theta} &= 0, & (6.29) \\ \mathcal{I}_1 + \frac{1}{4\pi}\sin\theta \frac{d\psi_0^{(0)}}{d\theta} &= 0, & (6.30) \\ L_\theta \psi_0^{(0)} &= 0, & (6.31) \end{aligned}$$

where from Eqs. (6.5), (6.6), and (6.20),

$$\mathcal{S}_2 = 16\pi^2 \mathcal{I}_1 \frac{d\mathcal{I}_1}{d\psi}. \tag{6.32}$$

We can easily integrate Eq. (6.31) and obtain

$$\psi_0^{(0)} = C_0^{(0)}(1-\cos\theta), \tag{6.33}$$

where $C_0^{(0)}$ is an integration constant. The above result implies that $\psi_0^{(0)}$ has only the monopole component. Then, substituting Eq. (6.33) into Eq. (6.30), we have

$$\mathcal{I}_1(X) = -\frac{1}{4\pi} C_0^{(0)} \hat{X}\left(2-\hat{X}\right), \tag{6.34}$$

where

$$\hat{X} = \frac{X}{C_0^{(0)}}. \tag{6.35}$$

It is non-trivial whether the lowest order of the inner light surface regularity condition (6.29) is satisfied by $\psi_0^{(0)}$ and $\mathcal{I}_1$ obtained above. From Eqs. (6.34) and (6.32), we have

$$\mathcal{S}_2(X) = 16\pi^2 \mathcal{I}_1(X) \frac{d\mathcal{I}_1}{dX}(X) = 2C_0^{(0)}\hat{X}\left(2 - 3\hat{X} + \hat{X}^2\right). \tag{6.36}$$

It is easy to check that Eqs. (6.33) and (6.36) satisfy Eq. (6.29). Namely, we have obtained a small electric current which satisfies the lowest order of the inner light surface regularity condition.

It is worthwhile to notice the meaning of the zeroth-order solutions for $\psi^{(0)}$, i.e., $\psi_0^{(0)}$. In the limit $\varepsilon \to 0$, $\Omega_{\rm F}$ and I become zero, whereas $\psi^{(0)}$ becomes $\psi_0^{(0)}$. Since the case $\Omega_{\rm F} = I = 0$ corresponds to the vacuum case, i.e., the case without an electric current, $\psi_0^{(0)}$ corresponds to the vacuum solution. As we showed in §V, the vacuum solution has only the monopole component on the event horizon. Eq. (6.33) is consistent with this fact. The small electric current $I = \varepsilon\mathcal{I}_1$ can be regarded as a result of the slowly rotating monopole field $\psi_0^{(0)}$.

2. *First-Order Solutions for* $\psi^{(0)}$

Next, we consider the correction of $O(\varepsilon^1)$ to the zeroth-order solution for $\psi^{(0)}$. Hereafter, we assume $C_0^{(0)} \neq 0$. In our perturbative method, the case of $C_0^{(0)} = 0$ is quite different from the case $C_0^{(0)} \neq 0$. If we choose $C_0^{(0)} = 0$, we can obtain only the trivial solution $\psi^{(0)} = 0$ using our perturbative method. We prove this statement in Appendix D.

The equations determining $\psi_1^{(0)}$ are derived from Eqs. (6.13), (6.16), and (6.28) of $O(\varepsilon^1)$:

$$\begin{aligned}
&\frac{d\psi_1^{(0)}}{d\theta} - \frac{1}{\sin 2\theta}\left(\frac{d\mathcal{S}_2}{d\psi}\psi_1^{(0)} + \mathcal{S}_3\right) \\
&\qquad + \sin\theta\left(\frac{d\psi_0^{(1)}}{d\theta} - \frac{1}{\sin 2\theta}\frac{d\mathcal{S}_2}{d\psi}\psi_0^{(1)} - \tan\theta\psi_0^{(1)}\right) = 0, \qquad (6.37)
\end{aligned}$$

$$\frac{d\mathcal{I}_1}{d\psi}\psi_1^{(0)} + \mathcal{I}_2 + \frac{1}{4\pi}\sin\theta\frac{d\psi_1^{(0)}}{d\theta} = 0, \qquad (6.38)$$

$$L_\theta\psi_1^{(0)} = 0, \qquad (6.39)$$

where

$$\mathcal{S}_3 = 16\pi^2\left(\mathcal{I}_1\frac{d\mathcal{I}_2}{d\psi} + \mathcal{I}_2\frac{d\mathcal{I}_1}{d\psi}\right). \qquad (6.40)$$

We can see from Eq. (6.39) that $\psi_1^{(0)}$ is also the monopole solution. Thus, the first-order correction merely adds a constant of $O(\varepsilon^1)$ to the integration constant of the zeroth-order solution. As a result, without loss of generality, we may assume for the first-order solutions that

$$\psi_1^{(0)} = 0. \qquad (6.41)$$

From Eqs. (6.38), (6.40), and the above equation, we have

$$\mathcal{I}_2 = 0 = \mathcal{S}_3. \qquad (6.42)$$

We should check the inner light surface regularity condition (6.13). Since $\psi_1^{(0)} = \mathcal{I}_2 = \mathcal{S}_3 = 0$, Eq. (6.37) becomes

$$\frac{d\psi_0^{(1)}}{d\theta} - \frac{1}{\sin 2\theta}\frac{d\mathcal{S}_2}{d\psi}\psi_0^{(1)} - \tan\theta\psi_0^{(1)} = 0. \qquad (6.43)$$

In order to estimate the above equation, we need $\psi_0^{(1)}$, which is the zeroth-order solution for $\psi^{(1)}$. We obtain $\psi_0^{(1)}$ from Eq. (6.26) of $O(\varepsilon^0)$ as

$$\psi_0^{(1)} = C'\sin\theta\frac{d\psi_0^{(0)}}{d\theta}, \qquad (6.44)$$

where we assume that C' is the order of unity. Substituting Eq. (6.33) into the above equation, we obtain

$$\psi_0^{(1)} = C' C_0^{(0)} \sin^2\theta. \tag{6.45}$$

By substituting Eq. (6.45) into Eq. (6.43) and using the functional form of $\mathcal{S}_2$ given by Eq. (6.36), we can see that Eq. (6.43) is satisfied. Here it is worthwhile to notice that $\psi_0^{(1)}$, as well as $\psi_0^{(0)}$, necessarily corresponds to a vacuum solution. It is easy to check that Eq. (6.45) is consistent with Eq. (5.5).

3. *Second-Order Solutions for* $\psi^{(0)}$

Hereafter, we will make frequent use of Eqs. (6.29) and (6.43) without giving an explicit reference. Now, we consider the correction of $O(\varepsilon^2)$ to the zeroth-order solution for $\psi^{(0)}$. The equations determining $\psi_2^{(0)}$ can be obtained from Eqs. (6.13), (6.16), and (6.28) of $O(\varepsilon^2)$:

$$\begin{aligned}\frac{d\psi_2^{(0)}}{d\theta} - \frac{1}{\sin 2\theta}\left(\frac{d\mathcal{S}_2}{d\psi}\psi_2^{(0)} + \mathcal{S}_4\right) = -\sin^2\theta\Bigg[&\left(2\tan\theta - \frac{1}{2\sin 2\theta}\frac{d^2\mathcal{S}_2}{d\psi^2}\psi_0^{(1)}\right)\psi_0^{(1)} \\ &+ \frac{d\psi_0^{(2)}}{d\theta} - \frac{1}{\sin 2\theta}\frac{d\mathcal{S}_2}{d\psi}\psi_0^{(2)} - 2\tan\theta\,\psi_0^{(2)}\Bigg],\end{aligned} \tag{6.46}$$

$$\frac{d\mathcal{I}_1}{d\psi}\psi_2^{(0)} + \mathcal{I}_3 + \frac{1}{4\pi}\sin\theta\frac{d\psi_2^{(0)}}{d\theta} = 0, \tag{6.47}$$

$$\begin{aligned}L_\theta\psi_2^{(0)} = \sin^2\theta\Bigg[&L_\theta\psi_0^{(2)} + 2\psi_0^{(2)} + 2\cot\theta\left(\frac{d\psi_0^{(2)}}{d\theta} - \frac{1}{\sin 2\theta}\frac{d\mathcal{S}_2}{d\psi}\psi_0^{(2)}\right) \\ &+ \psi_0^{(1)}\left(2 - \frac{1}{2\sin^2\theta}\frac{d^2\mathcal{S}_2}{d\psi^2}\psi_0^{(1)}\right)\Bigg],\end{aligned} \tag{6.48}$$

where

$$\mathcal{S}_4 = 16\pi^2\left(\mathcal{I}_1\frac{d\mathcal{I}_3}{d\psi} + \mathcal{I}_3\frac{d\mathcal{I}_1}{d\psi}\right). \tag{6.49}$$

In order to derive the above equations, we have used $\psi_1^{(0)} = \mathcal{I}_2 = 0$ and $\psi_1^{(1)} = 0$ obtained by substituting $\psi_1^{(0)} = 0$ into Eq. (6.26).

It should be noted that $\psi_0^{(2)}$ appears in Eqs. (6.46) and (6.48). In order to determine

$\psi_0^{(2)}$, we use Eq. (6.27) of $O(\varepsilon^2)$:

$$L_\theta \psi_2^{(0)} + 2\cot\theta \left[\frac{d\psi_2^{(0)}}{d\theta} - \frac{1}{\sin 2\theta} \left(\frac{d\mathcal{S}_2}{d\psi} \psi_2^{(0)} + \mathcal{S}_4 \right) \right] = 0. \tag{6.50}$$

By substituting Eq. (6.46) into the above equation, we have

$$L_\theta \psi_2^{(0)} = 2\cot\theta \sin^2\theta \left[\left(2\tan\theta - \frac{1}{2\sin 2\theta} \frac{d^2\mathcal{S}_2}{d\psi^2} \psi_0^{(1)} \right) \psi_0^{(1)} + \frac{d\psi_0^{(2)}}{d\theta} - \frac{1}{\sin 2\theta} \frac{d\mathcal{S}_2}{d\psi} \psi_0^{(2)} - 2\tan\theta \psi_0^{(2)} \right]. \tag{6.51}$$

By subtracting the above equation from Eq. (6.48), we obtain the equation for $\psi_0^{(2)}$ as

$$L_\theta \psi_0^{(2)} + 6\psi_0^{(2)} - 2\psi_0^{(1)} = 0. \tag{6.52}$$

It is easily seen from Eq. (6.45) that $\psi_0^{(2)} = \psi_0^{(1)}/2$ is a particular solution for the above equation. Thus, the general solution of the above equation is expressed by a linear combination of this particular solution and general solutions of the following homogeneous equation:

$$L_\theta f + 6f = 0. \tag{6.53}$$

The general solution of Eq. (6.53) is given by

$$f = \sin\theta \left[c_{\mathrm{p}} {P_2}^1(\cos\theta) + c_{\mathrm{q}} {Q_2}^1(\cos\theta) \right], \tag{6.54}$$

where c_{p} and c_{q} are arbitrary constants, and ${P_2}^1$ and ${Q_2}^1$ are the associated Legendre functions of the first and second kinds with $l = 2$ and $m = 1$, respectively. From the boundary condition at $\theta = 0$, c_{q} must vanish. Hence, the most general solution of Eq. (6.52), which satisfies the boundary condition at $\theta = 0$, is given by

$$\psi_0^{(2)} = C_0^{(2)} \sin^2\theta \cos\theta + \frac{C' C_0^{(0)}}{2} \sin^2\theta, \tag{6.55}$$

where we have used ${P_2}^1 = (3/2)\sin 2\theta$, and $C_0^{(2)}$ is an arbitrary constant. Here it is worthwhile to notice that the above result also corresponds to a vacuum solution. It is easy to check that Eq. (6.55), as well as $\psi_0^{(0)}$ and $\psi_0^{(1)}$, is consistent with Eq. (5.5).

By using Eq. (6.36) and substituting Eq. (6.45) into Eq. (6.48), we have

$$L_\theta \psi_2^{(0)} = \left[3C_0^{(0)} C' (1 + 2C' \cos\theta) - 4C_0^{(2)} \cos\theta \right] \sin^4\theta. \tag{6.56}$$

The above equation implies that, in general, the magnetic field on the event horizon includes non-monopole components of $O(\varepsilon^2)$.

We can easily integrate this equation and obtain

$$\psi_2^{(0)} = \frac{\sin^2\theta}{60}\left[2\left(3C^{(0)}C'^2 - 2C_0^{(2)}\right)\cos\theta\left(3\cos^2\theta - 7\right) + 15C_0^{(0)}C'\left(\cos^2\theta - 5\right)\right], \tag{6.57}$$

where we have chosen the integration constant so that $\psi_2^{(0)}|_{\theta=0} = \psi_2^{(0)}|_{\theta=\pi} = 0$, i.e., this correction consists of only non-monopole components.

The functional form of $\mathcal{I}_3$ is determined by using Eq. (6.47) as

$$\begin{aligned}\mathcal{I}_3(X) = &\frac{1}{120\pi}\hat{X}^2(2-\hat{X})^2 \\ &\times\left[15C_0^{(0)}C'\left(1-\hat{X}\right) + \left(3C_0^{(0)}C'^2 - 2C_0^{(2)}\right)\left(2 - 18\hat{X} + 9\hat{X}^2\right)\right],\end{aligned} \tag{6.58}$$

where $\hat{X}$ is defined by Eq. (6.35).

4. Solution near the inner light surface with corrections up to $O(\varepsilon^2)$

The solution with the corrections up to $O(\varepsilon^2)$ behaves near the inner light surface as

$$\psi = \psi_0^{(0)} + \varepsilon\psi_1^{(0)} + \varepsilon^2\psi_2^{(0)} + \left(\psi_0^{(1)} + \varepsilon\psi_1^{(1)}\right)(y-1) + \psi_0^{(2)}(y-1)^2 + \cdots. \tag{6.59}$$

Although the solutions for $\psi_N^{(n)}$ for $(n, N) = (0,0), (0,1), (0,2), (1,0), (1,1), (2,0)$ have been derived in the previous sections, we again show them with a slightly different parameterization:

$$\psi_0^{(0)} = C_0^{(0)}(1-\cos\theta), \tag{6.60}$$

$$\psi_1^{(0)} = 0, \tag{6.61}$$

$$\psi_2^{(0)} = \left[C_2^{(0)}\cos\theta\left(3\cos^2\theta - 7\right) + \frac{1}{4}C_0^{(0)}C'\left(\cos^2\theta - 5\right)\right]\sin^2\theta, \tag{6.62}$$

$$\psi_0^{(1)} = C_0^{(0)}C'\sin^2\theta, \tag{6.63}$$

$$\psi_1^{(1)} = 0, \tag{6.64}$$

$$\psi_0^{(2)} = \frac{1}{2}\left[3\left(C_0^{(0)}C'^2 - 10C_2^{(0)}\right)\cos\theta + C_0^{(0)}C'\right]\sin^2\theta, \tag{6.65}$$

where $C_0^{(2)}$ is given in this parameterization as

$$C_0^{(2)} = \frac{3}{2}\left(C_0^{(0)}C'^2 - 10C_2^{(0)}\right). \tag{6.66}$$

By using the same parameterization as the above, the electric current is given by

$$\begin{aligned}\mathcal{I}(X) = &-\frac{1}{8\pi}\hat{X}\left(2-\hat{X}\right)\left[2C_0^{(0)}\right. \\ &\left. - \varepsilon^2\hat{X}\left(2-\hat{X}\right)\left\{C_0^{(0)}C'(1-\hat{X}) + 2C_2^{(0)}\left(2 - 18\hat{X} + 9\hat{X}^2\right)\right\}\right],\end{aligned} \tag{6.67}$$

where $\hat{X}$ is defined by Eq. (6.35). We see that the arbitrary constants are only $C_N^{(0)}$ $(N = 0, 2)$ and C' . The reason this result takes this form is because if we choose ψ and $\partial_y \psi$ on the event horizon such that the regularity conditions (6.16) and (6.17) are satisfied, then ψ and $\mathcal{I}$ are completely determined.

VII. SUMMARY AND DISCUSSION

We studied a force-free magnetosphere in a static spherically symmetric black hole spacetime with a degenerate event horizon. We have found that if an electric current exists, higher multipole components of the magnetic field can be superposed upon the monopole component on the event horizon even if the two horizons degenerate into one horizon. This result is consistent with the numerical result given by Komissarov and McKinney: they showed that the magnetic field lines of higher multipole components can penetrate an extreme Kerr black hole if conductivity exists. The detailed geometrical structures of the extreme Kerr black hole and the extreme Reissner-Nordström black hole are different from each other. However, since the degenerate structures of the horizons of these black holes are similar, the present results may be applicable to a certain extent for the extreme Kerr black hole.

If we require that there is no monopole component in the lowest-order configuration on the horizon, or equivalently, $\psi_0^{(0)} = 0$, we obtain the trivial solution $\psi^{(0)} = 0$, even though we take all-order corrections into account (see Appendix D). Thus, the proposition in Appendix D seems to imply that there is no non-trivial configuration without a monopole component on the event horizon of the extreme Reissner-Nordström black hole, even if an electric current exists. But this is not necessarily true. In order to see this fact, note that there is an exact monopole solution for the Grad-Shafranov equation (3.6):

$$\Psi = C(1 - \cos\theta), \tag{7.1}$$

with the electric current

$$I = -\frac{\Omega_{\mathrm{F}}}{4\pi C}\Psi\left(2C - \Psi\right), \tag{7.2}$$

where C is an arbitrary constant. By contrast, the proposition in Appendix D implies that, if $\psi_0^{(0)} = 0$, there is no higher-order correction by which the configuration of the perturbative solution on the event horizon approaches to the monopole configuration in our perturbation scheme. In other words, our perturbative solution with vanishing $\psi_0^{(0)}$ cannot approach

to the above exact solution, even if we take into account all-order corrections. This fact suggests that even if an exact solution with a non-monopole configuration on the event horizon exists, the perturbative solution with vanishing $\psi_0^{(0)}$ cannot approach to such a solution in our perturbation scheme. This possibility may arise from the assumption for the electric current (6.20), which may be too strong, though the present analytic perturbation studies are impossible without this assumption.

We would like to stress again that it is very difficult to obtain a stationary force-free magnetosphere by solving the Grad-Shafranov equation for the extreme black hole spacetime numerically. Thus, we need to invoke analytic methods, as in the present study, or numerical techniques to follow the dynamical evolution of a force-free Maxwell field until a stationary configuration is realized, as Komissarov and McKinney used. As discussed in this paper, in the case that there are two light surfaces in the extreme Reissner-Nordström black hole spacetime, even though the magnetic field is regular, both on the event horizon and inner light surface, it will be singular on the outer light surface. If the angular velocity of the magnetic field is constant, two light surfaces necessarily exist. Thus, if the dynamical evolution of a force-free Maxwell field can be followed until it becomes stationary, then it is expected that the angular velocity decays far from the black hole so that the outer light surface does not exist. The extremity of charge or angular momentum changes the structure of boundary conditions for the Grad-Shafranov equation and seems to strongly affect global structures of the black hole magnetosphere.

Finally, we would like to suggest that analytic solutions obtained by this perturbation scheme becomes a benchmark for a numerical scheme to obtain solutions for stationary configurations of astrophysical magnetospheres, since our perturbation scheme is also suitable for non-extreme black hole cases.

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Appendix A: Derivation of Grad-Shafranov equation

A stationary, axisymmetric electromagnetic field implies $\partial_t A_a = 0 = \partial_\varphi A_a$. Then, from Eq. (3.3) and the force-free condition (3.4), we have $F_{t\theta}/F_{\theta\varphi} = F_{tr}/F_{r\varphi}$. Using these equations, the components of F_{ab} in the static coordinate system (2.1) are written in the form

$$F_{\mu\nu} = \begin{pmatrix} 0 & 0 & \Omega_{\rm F}\partial_r A_\varphi & \Omega_{\rm F}\partial_\theta A_\varphi \\ 0 & 0 & -\partial_r A_\varphi & -\partial_\theta A_\varphi \\ -\Omega_{\rm F}\partial_r A_\varphi & \partial_r A_\varphi & 0 & \sqrt{\gamma}B^\varphi \\ -\Omega_{\rm F}\partial_\theta A_\varphi & \partial_\theta A_\varphi & -\sqrt{\gamma}B^\varphi & 0 \end{pmatrix}, \tag{A1}$$

where $\Omega_{\rm F}$ is defined by Eq. (3.5), and

$$\gamma := \frac{r^6 \sin^2\theta}{\Delta} \tag{A2}$$

is the determinant of the intrinsic metric of the spacelike hypersurface labeled by t.

Note that $\Omega_{\rm F}$ can be regarded as the angular velocity of the magnetic field. We consider an observer with an angular velocity $d\varphi/dt = \Omega_{\rm F}$. His or her 4-velocity is given by $u^\mu = \Gamma(1, \Omega_{\rm F}, 0, 0)$, where Γ is a normalization factor. The electric field for this observer is given by $E_a = F_{ab}u^a$, and we can easily see from Eq. (A1) that E_a vanishes. Thus we may say that this observer is co-moving with the magnetic field, and the angular velocity of the magnetic field is $\Omega_{\rm F}$.

Substituting Eq. (A1) into the Jacobi identity $\partial_{[a}F_{bc]} = 0$, we have

$$(\partial_r\Omega_{\rm F})\partial_\theta A_\varphi - (\partial_\theta\Omega_{\rm F})\partial_r A_\varphi = 0. \tag{A3}$$

The above equation implies that $\partial_a\Omega_{\rm F} \propto \partial_a A_\varphi$, or equivalently, the equi-$\Omega_{\rm F}$ surface agrees with the equi-A_φ surface. Thus we have

$$\Omega_{\rm F} = \Omega_{\rm F}(A_\varphi). \tag{A4}$$

Using Eq. (A1), the Maxwell equations imply the following equations: the t-component implies

$$\partial_r\left(r^2\Omega_{\rm F}\sin\theta\,\partial_r A_\varphi\right) + \partial_\theta\left(\frac{r^2\Omega_{\rm F}\sin\theta}{\Delta}\partial_\theta A_\varphi\right) = -4\pi\alpha\sqrt{\gamma}J^t; \tag{A5}$$

the φ-component implies

$$\partial_r\left(\frac{\alpha^2}{\sin\theta}\partial_r A_\varphi\right) + \partial_\theta\left(\frac{\alpha^2}{\Delta\sin\theta}\partial_\theta A_\varphi\right) = -4\pi\alpha\sqrt{\gamma}J^\varphi; \tag{A6}$$

the r-component implies

$$\partial_\theta(\alpha B_\varphi) = 4\pi\alpha\sqrt{\gamma}J^r; \tag{A7}$$

and the θ-component implies

$$\partial_r(\alpha B_\varphi) = -4\pi\alpha\sqrt{\gamma}J^\theta, \tag{A8}$$

where

$$B_\varphi = r^2\sin^2\theta B^\varphi. \tag{A9}$$

Using Eq. (A1), the force-free condition implies

$$J^r\partial_r A_\varphi + J^\theta\partial_\theta A_\varphi \;=\; 0, \tag{A10}$$

$$(J^\varphi - J^t\Omega_{\rm F})\partial_r A_\varphi + \sqrt{\gamma}B^\varphi J^\theta \;=\; 0, \tag{A11}$$

$$(J^\varphi - J^t\Omega_{\rm F})\partial_\theta A_\varphi - \sqrt{\gamma}B^\varphi J^r \;=\; 0. \tag{A12}$$

Substituting Eq. (A8) to Eq. (A11), and substituting Eq. (A7) to Eq. (A12), we have

$$(J^\varphi - J^t\Omega_{\rm F})\partial_r A_\varphi - \frac{1}{4\pi\alpha}B^\varphi\partial_r(\alpha B_\varphi) \;=\; 0, \tag{A13}$$

$$(J^\varphi - J^t\Omega_{\rm F})\partial_\theta A_\varphi - \frac{1}{4\pi\alpha}B^\varphi\partial_\theta(\alpha B_\varphi) \;=\; 0. \tag{A14}$$

From the above equations, we have

$$\partial_r(\alpha B_\varphi)\partial_\theta A_\varphi - \partial_\theta(\alpha B_\varphi)\partial_r A_\varphi = 0. \tag{A15}$$

The above equations imply

$$\alpha B_\varphi = \mathcal{B}(A_\varphi). \tag{A16}$$

Using the above equation, Eqs. (A13) and (A14) imply

$$J^\varphi - J^t\Omega_{\rm F} = \frac{1}{4\pi\alpha^2 r^2\sin^2\theta}\mathcal{B}\frac{d\mathcal{B}}{dA_\varphi}. \tag{A17}$$

From Eqs. (A5) and (A6), we have

$$\begin{aligned}
&\partial_r\left(\frac{D\partial_r A_\varphi}{\sin\theta}\right) + \frac{1}{\Delta}\partial_\theta\left(\frac{D\partial_\theta A_\varphi}{\sin\theta}\right) + \frac{r^2\Omega_{\rm F}\sin\theta}{\Delta}\left[\Delta(\partial_r A)\partial_r\Omega_{\rm F} + (\partial_\theta A)\partial_\theta\Omega_{\rm F}\right] \\
&= -4\pi\alpha\sqrt{\gamma}\left(J^\varphi - J^t\Omega_{\rm F}\right),
\end{aligned} \tag{A18}$$

where D is defined by Eq. (3.10). Noting that $\Omega_{\rm F}$ is a function of A_φ and substituting Eq. (A17) into the right hand side of Eq. (A18), we have

$$\partial_r^2 A_\varphi + \frac{1}{\Delta}\left(L_\theta A_\varphi + \frac{N}{D}\right) = 0, \tag{A19}$$

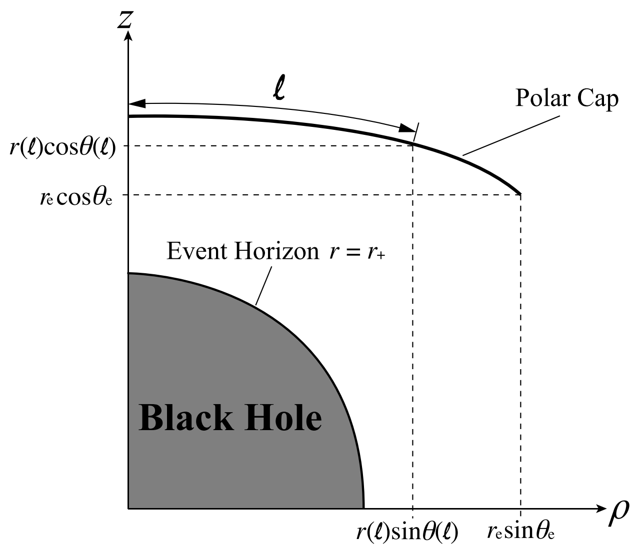

FIG. 3: The schematic diagram of a polar cap which is an axisymmetric two-dimensional spacelike surface parameterized by the proper length ℓ measured from the symmetry axis $\theta = 0$ along the polar cap.

where

$$L_\theta A_\varphi := \sin\theta \partial_\theta \left(\frac{\partial_\theta A_\varphi}{\sin\theta} \right), \tag{A20}$$

and

$$N := \Delta(\partial_r A_\varphi)\partial_r D + (\partial_\theta A_\varphi)\partial_\theta D + \frac{r^2}{2}\sin^2\theta \frac{d\Omega_{\mathrm{F}}^2}{dA_\varphi}\left[\Delta(\partial_r A_\varphi)^2 + (\partial_\theta A_\varphi)^2\right] + \frac{r^2}{2}\frac{d\mathcal{B}^2}{dA_\varphi}. \tag{A21}$$

The above equation is called the Grad-Shafranov equation.

Appendix B: Electric current and magnetic flux

Here we introduce electric current I and magnetic flux Ψ on a spacelike hypersurface labeled by t which penetrate downward and upward an axisymmetric polar cap, respectively. These quantities were first introduced by Macdonald and Thorne [8] and are related to $\mathcal{B}$ and A_φ as follows. The polar cap is parameterized by ℓ and φ, where ℓ is the proper length on the polar cap from $\theta = 0$. The coordinates r and θ on the polar cap are given as functions of ℓ, i.e., $r = r(\ell)$ and $\theta = \theta(\ell)$: by definition, $\theta(0) = 0$, and we assume that $r(0) > r_+$ (see

Fig. 3). The orthonormal tangent vectors of the polar cap are

$$e_{(\ell)}{}^{i} = \left(0,\ \frac{dr}{d\ell},\ \frac{d\theta}{d\ell}\right), \tag{B1}$$

$$e_{(\varphi)}{}^{i} = \left(\frac{1}{r\sin\theta},\ 0,\ 0\right). \tag{B2}$$

Then, the upward unit normal to the polar cap is

$$n_i = \frac{r^2}{\sqrt{\Delta}}\left(0,\ \frac{d\theta}{d\ell},\ \ -\frac{dr}{d\ell}\right). \tag{B3}$$

We assume that the edge of the polar cap is $r = r_{\rm e}$ and $\theta = \theta_{\rm e}$. Then, denoting the proper length ℓ at the edge by $\ell_{\rm e}$, we have

$$\begin{aligned} I &= -\int_0^{2\pi}\int_0^{\ell_{\rm e}} \alpha J^i n_i r\sin\theta d\ell d\varphi = -\frac{1}{2}\int_0^{\ell_{\rm e}}\left[\frac{d\theta}{d\ell}\partial_\theta(\alpha B_\varphi) + \frac{dr}{d\ell}\partial_r(\alpha B_\varphi)\right]d\ell \\ &= -\frac{1}{2}\int_0^{\ell_{\rm e}}\frac{d(\alpha B_\varphi)}{d\ell}d\ell = -\frac{1}{2}\alpha B_\varphi\Big|_{(r,\theta)=(r_{\rm e},\theta_{\rm e})}, \end{aligned} \tag{B4}$$

where we have used Eqs. (A7) and (A8) in the second equality and assumed that $B_\varphi|_{\theta=0} = 0$ from the regularity requirement. We can easily see from the above equation that the electric current through the polar cap is a function of the coordinate values of the edge, $(r_{\rm e}, \theta_{\rm e})$. By a similar consideration to that for the electric current I, the magnetic flux Ψ can be written in the form

$$\begin{aligned} \Psi &= \int_0^{2\pi}\int_0^{\ell_{\rm e}}\frac{1}{2}\epsilon^{ijk}F_{jk}n_i r\sin\theta d\ell d\varphi = \int_0^{2\pi}\int_0^{\ell_{\rm e}}\frac{1}{\sqrt{\gamma}}\left(F_{\theta\varphi}\frac{d\theta}{d\ell} - F_{\varphi r}\frac{dr}{d\ell}\right)\frac{r^3\sin\theta}{\sqrt{\Delta}}d\ell d\varphi \\ &= 2\pi\int_0^{\ell_{\rm e}}\left[\frac{d\theta}{d\ell}\partial_\theta A_\varphi + \frac{dr}{d\ell}\partial_r A_\varphi\right]d\ell = 2\pi\int_0^{\ell_{\rm e}}\frac{dA_\varphi}{d\ell}d\ell = 2\pi A_\varphi|_{(r,\theta)=(r_{\rm e},\theta_{\rm e})}, \end{aligned} \tag{B5}$$

where ϵ^{ijk} ($\epsilon^{\varphi r\theta} = 1/\sqrt{\gamma}$) is the components of the skew tensor in the spacelike hypersurface labeled by t, and we have assumed $A_\varphi|_{\theta=0} = 0$ from a regularity requirement. Thus we have

$$I = -\frac{1}{2}\mathcal{B} \qquad \text{and} \qquad \Psi = 2\pi A_\varphi. \tag{B6}$$

Rewriting Eq. (A19) using I and Ψ, the Grad-Shafranov equation Eq. (3.6) is obtained.

Appendix C: Relation between the Kerr-Schild and Static coordinate systems

The line element of the Reisner-Nordström spacetime with the Kerr-Schild coordinate system (T, Φ, R, Θ) is given by

$$
\begin{aligned}
ds^2 &= -\frac{R^2}{R^2+2MR-Q^2}dT^2 + R^2\sin^2\Theta d\Phi^2 \\
&+ \frac{R^2+2MR-Q^2}{R^2}\left(dR + \frac{2MR-Q^2}{R^2+2MR-Q^2}dT\right)^2 + R^2 d\Theta^2,
\end{aligned} \tag{C1}
$$

where $M = r_+ + r_-$ and $Q^2 = r_+ r_-$. The relation between the Kerr-Schild coordinate system and the static one is given by

$$dT = dt + \frac{2Mr-Q^2}{\Delta}dr, \tag{C2}$$

$$d\Phi = d\varphi, \tag{C3}$$

$$dR = dr, \tag{C4}$$

$$d\Theta = d\theta. \tag{C5}$$

From the above relation, we have $\Phi = \varphi$, $R = r$, and $\Theta = \theta$, and

$$\frac{\partial}{\partial T} = \frac{\partial}{\partial t}, \tag{C6}$$

$$\frac{\partial}{\partial \Phi} = \frac{\partial}{\partial \varphi}, \tag{C7}$$

$$\frac{\partial}{\partial R} = -\frac{2Mr-Q^2}{\Delta}\frac{\partial}{\partial t} + \frac{\partial}{\partial r}, \tag{C8}$$

$$\frac{\partial}{\partial \Theta} = \frac{\partial}{\partial \theta}. \tag{C9}$$

Using the above relations, we have

$$A_\varphi = A_\Phi. \tag{C10}$$

By virtue of the stationary, axisymmetric nature of the electromagnetic field, we can easily see that the components of F_{ab} in the Kerr-Schild coordinate system are given as

$$F_{T\Phi} = 0, \qquad F_{TR} = \Omega_{\rm F}\partial_r A_\varphi, \qquad F_{T\Theta} = \Omega_{\rm F}\partial_\theta A_\varphi, \tag{C11}$$

$$F_{\Phi R} = -\partial_r A_\varphi, \qquad F_{\Phi\Theta} = -\partial_\theta A_\varphi, \tag{C12}$$

$$
\begin{aligned}
F_{R\Theta} &= \sqrt{\gamma}B^\varphi - \frac{1}{\Delta}(2Mr-Q^2)\Omega_{\rm F}\partial_\theta A_\varphi \\
&= \frac{r^2}{\Delta\sin\theta}\left[\mathcal{B} - \frac{1}{r^2}(2Mr-Q^2)\Omega_{\rm F}\sin\theta\partial_\theta A_\varphi\right].
\end{aligned} \tag{C13}
$$

$$\tag{C14}$$

It should be noted that all the ordinary derivatives of the Kerr-Schild coordinates are

equivalent to those of the static coordinates for the stationary axisymmetric field $A_\Phi = A_\varphi$,

$$\begin{aligned}\partial_T A_\Phi &= \partial_t A_\varphi = 0, \quad \partial_\Phi A_\Phi = \partial_\varphi A_\varphi = 0, \\ \partial_R A_\Phi &= \partial_r A_\varphi, \quad \partial_\Theta A_\Phi = \partial_\theta A_\varphi, \quad \partial_R^2 A_\Phi = \partial_r^2 A_\varphi, \quad \text{and} \quad \partial_\Theta^2 A_\Phi = \partial_\theta^2 A_\varphi. \end{aligned} \tag{C15}$$

Thus, even if the Kerr-Schild coordinate system is adopted, the equation for A_Φ takes exactly the same form as Eq. (A19).

Appendix D: Non-existence of non-monopole solution

Proposition: Within the perturbation scheme developed in this paper, the solution for $\psi^{(0)}$ with vanishing lowest-order solution $\psi_0^{(0)}$ is the only trivial solution $\psi^{(0)} = 0$.

Proof. We prove this by induction. We have already shown that, if we impose the non-existence of a monopole component for the lowest order of the perturbative solution, we obtain

$$\psi_0^{(0)} = 0. \tag{D1}$$

Here, we assume that $\psi_N^{(0)} = 0$ for $0 \le N \le M$, or equivalently,

$$\psi^{(0)}(\theta) = \varepsilon^{M+1} \sum_{N=0}^{\infty} \varepsilon^N \psi_{N+M+1}(\theta). \tag{D2}$$

Then, we have

$$\mathcal{I}(\psi^{(0)}) = \sum_{N=0}^{\infty} \varepsilon^N \mathcal{I}_N \left(\varepsilon^{M+1} \sum_{J=0}^{\infty} \varepsilon^J \psi_{J+M+1} \right) = \varepsilon^{M+1} \mathcal{I}_0{}' \psi_{M+1}^{(0)} + \mathcal{O}(\varepsilon^{M+2}), \tag{D3}$$

where

$$\mathcal{I}_0{}' := \left. \frac{d\mathcal{I}_0(\psi)}{d\psi} \right|_{\psi=0}, \tag{D4}$$

and we have used $\mathcal{I}_N(0) = 0$, which is required from the regularity condition at $\theta = 0$. Substituting the above equation into Eq. (6.16), we obtain an equation of $O(\varepsilon^{M+1})$,

$$\mathcal{I}_0{}' \psi_{M+1}^{(0)} + \frac{1}{4\pi} \sin\theta \frac{d\psi_{M+1}^{(0)}}{d\theta} = 0. \tag{D5}$$

Integrating the above equation, we have

$$\psi_{M+1}^{(0)} = C_{M+1}^{(0)} \left[\frac{\sin^2\theta}{(1+\cos\theta)^2} \right]^{-2\pi\mathcal{I}_0{}'}, \tag{D6}$$

where $C^{(0)}_{M+1}$ is an integration constant. The regularity condition implies $\psi^{(0)}_{M+1}|_{\theta=0} = 0 = d\psi^{(0)}_{M+1}/d\theta|_{\theta=0}$. Hence, if $C^{(0)}_{M+1}$ does not vanish,

$$-2\pi\mathcal{I}_0{}' = 1 + c^2 \tag{D7}$$

must be satisfied, where c is an arbitrary constant.

In the neighborhood of $\theta = \pi$, $\sin\theta \sim \pi - \theta$ and $\cos\theta \sim -1 + (\theta - \pi)^2/2$. Hence, if $C^{(0)}_{M+1}$ does not vanish, we have

$$\lim_{\theta\to\pi} \psi^{(0)}_{M+1} = \infty. \tag{D8}$$

If we require the finiteness of $\psi^{(0)}_{M+1}$ at $\theta = \pi$, then $C^{(0)}_{M+1}$ must vanish, and, as a result, $\psi^{(0)}_{M+1} = 0$ is obtained. Q.E.D.

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